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An irreducible approach to second-order reducible second-class constraints

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Abstract

An irreducible canonical approach to second-order reducible second-class constraints is given. The procedure is exemplified on gauge-fixed 3-forms.

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1. Introduction

The canonical approach to systems with reducible second-class constraints is quite intricate, demanding a modification of the usual rules as the matrix of the Poisson brackets among the constraints is not invertible. Thus, it is necessary to isolate a set of independent constraints and then construct the Dirac bracket [1, 2] with respect to this set. The split of the constraints may lead to the loss of important symmetries, so it should be avoided. As shown in [3–8], it is however possible to construct the Dirac bracket in terms of a noninvertible matrix without separating the independent constraint functions. A third possibility is to substitute the reducible second-class constraints by some irreducible ones and further work with the Dirac bracket based on the irreducible constraints. This idea, suggested in [9] mainly in the context of 2- and 3-form gauge fields, has been developed in a general manner only for first-order reducible second-class constraints [10].

In this paper, we give an irreducible approach to second-order reducible second-class constraints. Our strategy includes three main steps. First, we express the Dirac bracket for the reducible system in terms of an invertible matrix. Second, we construct an intermediate second-order reducible second-class system on a larger phase space and establish the equality between the original Dirac bracket and that corresponding to the intermediate theory. Third, we prove that there exists an irreducible second-class constraint set equivalent to the intermediate one, such that the corresponding Dirac brackets coincide. These three steps enforce the fact that the fundamental Dirac brackets derived within the irreducible and original reducible settings coincide.

The present paper is organized in five sections. In section 2, we briefly review the procedure for first-order reducible second-class constraints. Section 3 is the 'hard core' of the paper. Here, we approach second-order reducible second-class constraints by implementing the three main steps mentioned above. In section 4, we exemplify in detail the general procedure from section 3 in the case of gauge-fixed three-form gauge fields. Section 5 ends the paper with the main conclusions.

2. First-order reducible second-class constraints: a brief review

2.1. Dirac bracket for first-order reducible second-class constraints

We start with a system locally described by N canonical pairs $z^a = (q^i, p_i)$, subject to some constraints

$$\chi_{\alpha_0}(z^a) \approx 0, \qquad \alpha_0 = 1, \ldots, M_0. \tag{1}$$

For simplicity, we take all the phase-space variables to be bosonic. However, our analysis can be extended to fermionic degrees of freedom modulo including some appropriate phase factors. We choose the scenario of systems with a finite number of degrees of freedom only for notational simplicity, but our approach is equally valid for field theories. In addition, we presume that the functions χ_{α_0} are not all independent, but there exist some nonvanishing functions $Z^{\alpha_0}_{\alpha_1}$ such that

$$Z^{\alpha_0}_{\alpha_1}\chi_{\alpha_0} = 0, \qquad \alpha_1 = 1, \ldots, M_1. \tag{2}$$

Moreover, we assume that $Z^{\alpha_0}_{\alpha_1}$ are all independent and (2) are the only reducibility relations with respect to the constraints (1). These constraints are purely second class if any maximal, independent set of $M_0 - M_1$ constraint functions χ_A $(A = 1, \ldots, M_0 - M_1)$ among χ_{α_0} is such that the matrix

$$C^{(1)}_{AB} = [\chi_A, \chi_B] \tag{3}$$

is invertible. Here and in the following, the symbol [,] denotes the Poisson bracket. In terms of independent constraints, the Dirac bracket takes the form

$$[F, G]^{(1)*} = [F, G] - [F, \chi_A] M^{(1)AB} [\chi_B, G], \tag{4}$$

where $M^{(1)AB} C^{(1)}_{BC} \approx \delta^A_C$. In the previous relations we introduced an extra index, (1), having the role to emphasize that the Dirac bracket (4) is based on a first-order reducible second-class constraint set. We can rewrite the Dirac bracket (4) without finding a definite subset of independent second-class constraints as follows. We start with the matrix

$$C^{(1)}_{\alpha_0\beta_0} = \left[\chi_{\alpha_0}, \chi_{\beta_0}\right], \tag{5}$$

which clearly is not invertible because

$$Z^{\alpha_0}_{\alpha_1} C^{(1)}_{\alpha_0\beta_0} \approx 0. \tag{6}$$

If $\bar{a}^{\alpha_1}_{\alpha_0}$ is a solution to the equation

$$\bar{a}^{\alpha_1}_{\alpha_0} Z^{\alpha_0}_{\beta_1} \approx \delta^{\alpha_1}_{\beta_1}, \tag{7}$$

then we can introduce a matrix [6] $M^{(1)\alpha_0\beta_0}$ through the relation

$$M^{(1)\alpha_0\beta_0} C^{(1)}_{\beta_0\gamma_0} \approx \delta^{\alpha_0}_{\gamma_0} - Z^{\alpha_0}_{\alpha_1} \bar{a}^{\alpha_1}_{\gamma_0} \equiv d^{\alpha_0}_{\gamma_0}, \tag{8}$$

with $M^{(1)\alpha_0\beta_0} = -M^{(1)\beta_0\alpha_0}$. Then, formula [6]

$$[F, G]^{(1)*} = [F, G] - \left[F, \chi_{\alpha_0}\right] M^{(1)\alpha_0\beta_0} \left[\chi_{\beta_0}, G\right] \tag{9}$$

defines the same Dirac bracket like (4) on the surface (1). We remark that there exist some ambiguities in defining the matrix $M^{(1)\alpha_0\beta_0}$ since if we make the transformation

$$M^{(1)\alpha_0\beta_0} \to M^{(1)\alpha_0\beta_0} + Z^{\alpha_0}_{\alpha_1} q^{\alpha_1\beta_1} Z^{\beta_0}_{\beta_1}, \tag{10}$$

with $q^{\alpha_1\beta_1}$ some completely antisymmetric functions, then (8) is still satisfied.

At this stage it is useful to make some comments. First, we remark that relations (7) and (8) yield

$$M^{(1)\alpha_0\beta_0} C^{(1)}_{\beta_0\gamma_0} Z^{\gamma_0}_{\beta_1} \approx 0, \tag{11}$$

which ensures the fact that the rank of $M^{(1)\alpha_0\beta_0} C^{(1)}_{\beta_0\gamma_0}$ is equal to the number of independent second-class constraints, i.e.

$$\operatorname{rank}\left(M^{(1)\alpha_0\beta_0} C^{(1)}_{\beta_0\gamma_0}\right) \approx M_0 - M_1. \tag{12}$$

Second, by means of (8) we deduce the relation

$$\left[\chi_{\alpha_0}, G\right]^{(1)*} \approx -\bar{a}^{\alpha_1}_{\alpha_0} \left[Z^{\beta_0}_{\alpha_1}, G\right] \chi_{\beta_0}, \tag{13}$$

which ensures

$$\left[\chi_{\alpha_0}, G\right]^{(1)*} = 0, \qquad \text{for any } G, \tag{14}$$

on the second-class surface, as required by the general properties of the Dirac bracket. Third, we remark that, in spite of the fact that the matrix $C^{(1)}_{\alpha_0\beta_0}$ is not invertible, the Dirac bracket expressed by (9) still satisfies Jacobi's identity

$$[[F, G]^{(1)*}, P]^{(1)*} + [[P, F]^{(1)*}, G]^{(1)*} + [[G, P]^{(1)*}, F]^{(1)*} \approx 0 \tag{15}$$

on surface (1). The proof follows the same line like in the irreducible case. Let

$$\bar{F} = F + u^{\beta_0} \chi_{\beta_0} \tag{16}$$

be a function such that

$$\left[\bar{F}, \chi_{\alpha_0}\right] \approx 0. \tag{17}$$

Thus, in order to construct $\bar{F}$ we must solve the equation

$$u^{\beta_0} C^{(1)}_{\beta_0\alpha_0} \approx -\left[F, \chi_{\alpha_0}\right]. \tag{18}$$

Based on

$$d^{\lambda_0}_{\alpha_0} \chi_{\lambda_0} = \chi_{\alpha_0}, \tag{19}$$

it follows in a simple manner that the solution to (18) is given by

$$u^{\beta_0} = \left[F, \chi_{\lambda_0}\right] M^{(1)\beta_0\lambda_0}, \tag{20}$$

which further leads to

$$\bar{F} = F + \left[F, \chi_{\beta_0}\right] M^{(1)\alpha_0\beta_0} \chi_{\alpha_0}. \tag{21}$$

Relying on (19) and (21), by direct computation we arrive at the relation

$$[[F, G]^{(1)*}, P]^{(1)*} \approx [[\bar{F}, \bar{G}], \bar{P}], \tag{22}$$

which indicates that identity (15) is ensured by Jacobi's identity corresponding to the Poisson bracket for the functions $\bar{F}$, $\bar{G}$ and $\bar{P}$. We mention that the key point of the proof of Jacobi's identity (15) is represented by relation (19).

2.2. Irreducible analysis of first-order reducible second-class constraints

First-order reducible second-class constraints can be approached in an irreducible manner, as has been shown in [10]. To this end, one starts from the solution to (7)

$$\bar{a}^{\alpha_1}_{\alpha_0} = \bar{D}^{\alpha_1}_{\gamma_1} a^{\gamma_1}_{\alpha_0}, \tag{23}$$

where $a^{\gamma_1}_{\alpha_0}$ are some functions chosen such that

$$\operatorname{rank}\left(Z^{\alpha_0}_{\alpha_1} a^{\gamma_1}_{\alpha_0}\right) = M_1 \tag{24}$$

and $\bar{D}^{\beta_1}_{\gamma_1}$ stands for the inverse of $Z^{\alpha_0}_{\alpha_1} a^{\gamma_1}_{\alpha_0}$. In order to develop an irreducible approach it is necessary to enlarge the original phase space with some new variables $\left(Y_{\alpha_1}\right)_{\alpha_1=1,\ldots,M_1}$, endowed with the Poisson brackets

$$\left[Y_{\alpha_1}, Y_{\beta_1}\right] = \Gamma_{\alpha_1\beta_1}, \tag{25}$$

where $\Gamma_{\alpha_1\beta_1}$ are the elements of an invertible, antisymmetric matrix that may depend on the newly added variables. Consequently, one constructs the constraints

$$\bar{\chi}_{\alpha_0} = \chi_{\alpha_0} + a^{\alpha_1}_{\alpha_0} Y_{\alpha_1} \approx 0, \tag{26}$$

which are second-class and, essentially, irreducible. Following the line exposed in [10], it can be shown that the Dirac bracket associated with the irreducible constraints takes the form

$$[F, G]^{(1)*}|_{\text{ired}} = [F, G] - \left[F, \bar{\chi}_{\alpha_0}\right] \mu^{(1)\alpha_0\beta_0} \left[\bar{\chi}_{\beta_0}, G\right], \tag{27}$$

and it is (weakly) equal to the original Dirac bracket (4),

$$[F, G]^{(1)*} \approx [F, G]^{(1)*}|_{\text{ired}}. \tag{28}$$

In (27) the quantities $\mu^{(1)\alpha_0\beta_0}$ are the elements of an invertible, antisymmetric matrix, expressed by

$$\mu^{(1)\alpha_0\beta_0} \approx M^{(1)\alpha_0\beta_0} + Z^{\alpha_0}_{\lambda_1} \bar{D}^{\lambda_1}_{\beta_1} \Gamma^{\beta_1\gamma_1} \bar{D}^{\sigma_1}_{\gamma_1} Z^{\beta_0}_{\sigma_1}, \tag{29}$$

with $\Gamma^{\beta_1\gamma_1}$ the inverse of $\Gamma_{\alpha_1\beta_1}$. Formula (28) is essential in our context because it proves that one can indeed approach first-order reducible second-class constraints in an irreducible fashion.

3. Second-order reducible second-class constraints

3.1. Reducible approach

3.1.1. Dirac bracket for second-order reducible second-class constraints. In the following, we will generalize the previous approach to the case of second-order reducible second-class constraints. This means that not all of the first-order reducibility functions $Z^{\alpha_0}_{\alpha_1}$ are independent. Beside the first-order reducibility relations (2), there appear also the second-order reducibility relations

$$Z^{\alpha_1}_{\alpha_2} Z^{\alpha_0}_{\alpha_1} \approx 0, \qquad \alpha_2 = 1, \ldots, M_2. \tag{30}$$

We will assume that the reducibility stops at order 2, so the functions $Z^{\alpha_1}_{\alpha_2}$ are by hypothesis taken to be independent. It is understood that $Z^{\alpha_1}_{\alpha_2}$'s define a complete set of reducibility functions for $Z^{\alpha_0}_{\alpha_1}$. In this situation, the number of independent second-class constraints is equal to $M_0 - M_1 + M_2$. As a consequence, we can work with a Dirac bracket of the type (4), but in terms of $M_0 - M_1 + M_2$ independent functions χ_A

$$[F, G]^{(2)*} = [F, G] - [F, \chi_A] M^{(2)AB} [\chi_B, G], \qquad A = 1, \ldots, M_0 - M_1 + M_2, \tag{31}$$

where $M^{(2)AB}C^{(2)}_{BC} \approx \delta^A_C$, with $C^{(2)}_{AB} = [\chi_A, \chi_B]$. It is obvious that the matrix

$$C^{(2)}_{\alpha_0\beta_0} = \left[\chi_{\alpha_0}, \chi_{\beta_0}\right] \tag{32}$$

satisfies the relations

$$Z^{\alpha_0}_{\alpha_1} C^{(2)}_{\alpha_0\beta_0} \approx 0, \tag{33}$$

so its rank is equal to $M_0 - M_1 + M_2$.

Let $\bar{A}^{\alpha_2}_{\alpha_1}$ be a solution of the equation

$$Z^{\alpha_1}_{\beta_2} \bar{A}^{\alpha_2}_{\alpha_1} \approx \delta^{\alpha_2}_{\beta_2} \tag{34}$$

and $\bar{\omega}_{\beta_1\gamma_1} = -\bar{\omega}_{\gamma_1\beta_1}$ a solution to

$$Z^{\beta_1}_{\beta_2} \bar{\omega}_{\beta_1\gamma_1} \approx 0. \tag{35}$$

We define an antisymmetric matrix $\hat{\omega}^{\alpha_1\beta_1}$ through the relation

$$\hat{\omega}^{\alpha_1\beta_1} \bar{\omega}_{\beta_1\gamma_1} \approx \delta^{\alpha_1}_{\gamma_1} - Z^{\alpha_1}_{\alpha_2} \bar{A}^{\alpha_2}_{\gamma_1} \equiv D^{\alpha_1}_{\gamma_1}. \tag{36}$$

Taking (35) into account, it results that $\hat{\omega}^{\alpha_1\beta_1}$ contains some ambiguities, namely it is defined up to the transformation

$$\hat{\omega}^{\alpha_1\beta_1} \rightarrow \hat{\omega}^{\alpha_1\beta_1} + Z^{\alpha_1}_{\alpha_2} q^{\alpha_2\beta_2} Z^{\beta_1}_{\beta_2}, \tag{37}$$

with $q^{\alpha_2\beta_2}$ some arbitrary, antisymmetric functions. On the other hand, simple computation shows that the matrix $D^{\alpha_1}_{\gamma_1}$ satisfies the properties

$$\bar{A}^{\alpha_2}_{\alpha_1} D^{\alpha_1}_{\gamma_1} \approx 0, \qquad Z^{\gamma_1}_{\gamma_2} D^{\alpha_1}_{\gamma_1} \approx 0, \tag{38}$$

$$Z^{\alpha_0}_{\alpha_1} D^{\alpha_1}_{\gamma_1} \approx Z^{\alpha_0}_{\gamma_1}, \qquad D^{\alpha_1}_{\gamma_1} D^{\gamma_1}_{\lambda_1} \approx D^{\alpha_1}_{\lambda_1}. \tag{39}$$

Based on the latter formula from (38) we infer an alternative expression for $D^{\alpha_1}_{\gamma_1}$, namely

$$D^{\alpha_1}_{\gamma_1} \approx \bar{A}^{\alpha_1}_{\alpha_0} Z^{\alpha_0}_{\gamma_1}, \tag{40}$$

for some functions $\bar{A}^{\alpha_1}_{\alpha_0}$. From the former relation in (39) and (40), we deduce that

$$Z^{\gamma_0}_{\gamma_1} D^{\alpha_0}_{\gamma_0} \approx 0, \tag{41}$$

where

$$D^{\alpha_0}_{\gamma_0} \approx \delta^{\alpha_0}_{\gamma_0} - Z^{\alpha_0}_{\alpha_1} \bar{A}^{\alpha_1}_{\gamma_0}. \tag{42}$$

At this stage, we can rewrite the Dirac bracket (31) without separating a specific subset of independent constraints. In view of this, we introduce an antisymmetric matrix $M^{(2)\alpha_0\beta_0}$ through the relation

$$M^{(2)\alpha_0\beta_0} C^{(2)}_{\beta_0\gamma_0} \approx D^{\alpha_0}_{\gamma_0}, \tag{43}$$

such that the formula

$$[F, G]^{(2)*} = [F, G] - \left[F, \chi_{\alpha_0}\right] M^{(2)\alpha_0\beta_0} \left[\chi_{\beta_0}, G\right] \tag{44}$$

defines the same Dirac bracket like (31) on the surface (1). It is simple to see that $M^{(2)\alpha_0\beta_0}$ also contains some ambiguities, being defined up to the transformation

$$M^{(2)\alpha_0\beta_0} \rightarrow M^{(2)\alpha_0\beta_0} + Z^{\alpha_0}_{\alpha_1} \hat{q}^{\alpha_1\beta_1} Z^{\beta_0}_{\beta_1}, \tag{45}$$

with $\hat{q}^{\alpha_1\beta_1}$ some antisymmetric, but otherwise arbitrary functions. Relations (30) and (41) ensure that

$$\operatorname{rank}\left(D^{\alpha_0}_{\gamma_0}\right) \approx M_0 - M_1 + M_2, \tag{46}$$

so the rank of $M^{(2)\alpha_0\beta_0}C^{(2)}_{\beta_0\gamma_0}$ is equal to the number of independent second-class constraints also in the presence of the second-order reducibility. At the same time, we have that

$$[\chi_{\alpha_0}, G]^{(2)*} \approx -\bar{A}^{\alpha_1}_{\alpha_0}[Z^{\beta_0}_{\alpha_1}, G]\chi_{\beta_0}, \tag{47}$$

so we recover the property $[\chi_{\alpha_0}, G]^{(2)*} = 0$ (for any G) on the surface of second-order reducible second-class constraints. The fact that the Dirac bracket given by (44) satisfies Jacobi's identity can be proved like in the first-order reducible case. The analogous of the key relation (19) from the first-order reducible situation is now $D^{\alpha_0}_{\gamma_0}\chi_{\alpha_0} = \chi_{\gamma_0}$.

3.1.2. Dirac bracket in terms of an invertible matrix. Before expressing the Dirac bracket in terms of an invertible matrix, we will analyze equations (34) and (35). The solution to (34) can be written as

$$\bar{A}^{\alpha_2}_{\alpha_1} \approx \bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\alpha_1}, \tag{48}$$

where $A^{\lambda_2}_{\alpha_1}$ are some functions chosen such that the matrix

$$D^{\lambda_2}_{\beta_2} = Z^{\alpha_1}_{\beta_2}A^{\lambda_2}_{\alpha_1} \tag{49}$$

is of maximum rank,

$$\mathrm{rank}\left(D^{\lambda_2}_{\beta_2}\right) = M_2, \tag{50}$$

with $\bar{D}^{\alpha_2}_{\lambda_2}$ the inverse of $D^{\lambda_2}_{\beta_2}$. (Strictly speaking, the solution to (34) has the general form $\bar{A}^{\alpha_2}_{\alpha_1} \approx \bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\alpha_1} + u^{\alpha_2}_{\alpha_0}Z^{\alpha_0}_{\alpha_1} + v^{\alpha_2\lambda_1}\bar{\omega}_{\lambda_1\alpha_1}$, where $u^{\alpha_2}_{\alpha_0}$ and $v^{\alpha_2\lambda_1}$ are arbitrary functions. By making the redefinitions $u^{\alpha_2}_{\alpha_0} = \bar{D}^{\alpha_2}_{\lambda_2}\hat{u}^{\lambda_2}_{\alpha_0}$ and $v^{\alpha_2\lambda_1} = \bar{D}^{\alpha_2}_{\lambda_2}\hat{v}^{\lambda_2\lambda_1}$, with $\hat{u}^{\lambda_2}_{\alpha_0}$ and $\hat{v}^{\lambda_2\lambda_1}$ arbitrary, we can set $\bar{A}^{\alpha_2}_{\alpha_1}$ in the form $\bar{A}^{\alpha_2}_{\alpha_1} \approx \bar{D}^{\alpha_2}_{\lambda_2}\left(A^{\lambda_2}_{\alpha_1} + \hat{u}^{\lambda_2}_{\alpha_0}Z^{\alpha_0}_{\alpha_1} + \hat{v}^{\lambda_2\lambda_1}\bar{\omega}_{\lambda_1\alpha_1}\right)$. On the other hand, the functions $A^{\lambda_2}_{\alpha_1}$ with the property that the rank of matrix (49) is maximum are defined up to the transformation $A^{\lambda_2}_{\alpha_1} \to A'^{\lambda_2}_{\alpha_1} = A^{\lambda_2}_{\alpha_1} + \tau^{\lambda_2}_{\alpha_0}Z^{\alpha_0}_{\alpha_1} + \lambda^{\lambda_2\lambda_1}\bar{\omega}_{\lambda_1\alpha_1}$, in the sense that $Z^{\alpha_1}_{\beta_2}A^{\lambda_2}_{\alpha_1} \approx Z^{\alpha_1}_{\beta_2}A'^{\lambda_2}_{\alpha_1}$, where $\tau^{\lambda_2}_{\alpha_0}$ and $\lambda^{\lambda_2\lambda_1}$ are also arbitrary. Thus, we can always absorb the quantity $\hat{u}^{\lambda_2}_{\alpha_0}Z^{\alpha_0}_{\alpha_1} + \hat{v}^{\lambda_2\lambda_1}\bar{\omega}_{\lambda_1\alpha_1}$ from $\bar{A}^{\alpha_2}_{\alpha_1}$ by redefining $A^{\lambda_2}_{\alpha_1}$, such that we finally obtain solution (48).) Then, on the one hand we have that

$$D^{\alpha_1}_{\gamma_1} \approx \delta^{\alpha_1}_{\gamma_1} - Z^{\alpha_1}_{\alpha_2}\bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\gamma_1} \tag{51}$$

and on the other hand (inserting (48) in the former relation from (38)) we can write

$$A^{\sigma_2}_{\alpha_1}D^{\alpha_1}_{\gamma_1} \approx 0. \tag{52}$$

Substituting (40) in (52), we are led to

$$\bar{A}^{\alpha_1}_{\alpha_0}A^{\alpha_2}_{\alpha_1} \approx 0, \tag{53}$$

which further implies

$$\bar{A}^{\gamma_1}_{\alpha_0}D^{\alpha_1}_{\gamma_1} \approx \bar{A}^{\alpha_1}_{\alpha_0}. \tag{54}$$

Based on the latter formula from (38), we find that the solution to (35) can be expressed as

$$\bar{\omega}_{\beta_1\gamma_1} \approx D^{\tau_1}_{\beta_1}\tilde{\omega}_{\tau_1\lambda_1}D^{\lambda_1}_{\gamma_1}, \tag{55}$$

where $\tilde{\omega}_{\tau_1\lambda_1}$ is antisymmetric. Acting with $A^{\alpha_2}_{\alpha_1}$ on (36) and taking into account (52) and (55), we reach the equation

$$A^{\alpha_2}_{\alpha_1}\hat{\omega}^{\alpha_1\beta_1}\bar{\omega}_{\beta_1\gamma_1} \approx 0, \tag{56}$$

whose solution can be chosen as

$$\hat{\omega}^{\alpha_1\beta_1} = D^{\alpha_1}_{\rho_1}\tilde{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1}, \tag{57}$$

with $\tilde{\omega}^{\rho_1\sigma_1}$ antisymmetric. (In fact, the general solution of (56) is given by $\hat{\omega}^{\alpha_1\beta_1} = D^{\alpha_1}_{\rho_1}\tilde{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1} + Z^{\alpha_1}_{\alpha_2}u^{\alpha_2\beta_2}Z^{\beta_1}_{\beta_2}$, with $u^{\alpha_2\beta_2}$ arbitrary, antisymmetric functions. Since $\hat{\omega}^{\alpha_1\beta_1}$ are defined up to transformation (37), we can always absorb the terms $Z^{\alpha_1}_{\alpha_2}u^{\alpha_2\beta_2}Z^{\beta_1}_{\beta_2}$ through a redefinition of $\hat{\omega}^{\alpha_1\beta_1}$ and finally arrive at (57).) With the help of (52) and (57), it is easy to see that

$$A^{\alpha_2}_{\alpha_1}\hat{\omega}^{\alpha_1\beta_1} \approx 0. \tag{58}$$

Except from being antisymmetric, the matrices $\tilde{\omega}_{\tau_1\lambda_1}$ and $\tilde{\omega}^{\rho_1\sigma_1}$ are arbitrary at this point. Nevertheless, they can be chosen to satisfy a series of useful properties, as the next theorem proves.

Theorem 1. *The matrices of elements $\tilde{\omega}_{\tau_1\lambda_1}$ and $\tilde{\omega}^{\rho_1\sigma_1}$ can always be taken to satisfy the following properties:*

(a) (weak) invertibility,
(b) fulfillment of relation

$$\tilde{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1}\tilde{\omega}_{\beta_1\lambda_1} \approx D^{\rho_1}_{\lambda_1}, \tag{59}$$

(c) (weak) mutual invertibility

$$\tilde{\omega}^{\rho_1\sigma_1}\tilde{\omega}_{\sigma_1\lambda_1} \approx \delta^{\rho_1}_{\lambda_1}. \tag{60}$$

Proof.

(a) Replacing the latter formula from (39) in (55) and (57), we infer the relations

$$D^{\tau_1}_{\beta_1}\bar{\omega}_{\tau_1\lambda_1}D^{\lambda_1}_{\gamma_1} \approx D^{\tau_1}_{\beta_1}\tilde{\omega}_{\tau_1\lambda_1}D^{\lambda_1}_{\gamma_1}, \tag{61}$$

$$D^{\alpha_1}_{\rho_1}\hat{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1} \approx D^{\alpha_1}_{\rho_1}\tilde{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1}, \tag{62}$$

with the help of which we further deduce

$$\tilde{\omega}_{\tau_1\lambda_1} \approx \bar{\omega}_{\tau_1\lambda_1} + \bar{D}^{\sigma_2}_{\tau_2}A^{\tau_2}_{\tau_1}\omega_{\sigma_2\gamma_2}A^{\lambda_2}_{\lambda_1}\bar{D}^{\gamma_2}_{\lambda_2}, \tag{63}$$

$$\tilde{\omega}^{\rho_1\sigma_1} \approx \hat{\omega}^{\rho_1\sigma_1} + Z^{\rho_1}_{\alpha_2}\omega^{\alpha_2\beta_2}Z^{\sigma_1}_{\beta_2}, \tag{64}$$

for some antisymmetric matrices $\omega_{\sigma_2\gamma_2}$ and $\omega^{\alpha_2\beta_2}$, taken to be invertible. Each of the terms from the right-hand sides of formulae (63) and (64) displays null vectors. The null vectors of $\bar{\omega}_{\tau_1\lambda_1}$ and $\hat{\omega}^{\rho_1\sigma_1}$ are $Z^{\lambda_1}_{\alpha_2}$ and $A^{\rho_2}_{\rho_1}$ respectively (see (35) and (58)), while the null vectors of $\bar{D}^{\sigma_2}_{\tau_2}A^{\tau_2}_{\tau_1}\omega_{\sigma_2\gamma_2}A^{\lambda_2}_{\lambda_1}\bar{D}^{\gamma_2}_{\lambda_2}$ and $Z^{\rho_1}_{\alpha_2}\omega^{\alpha_2\beta_2}Z^{\sigma_1}_{\beta_2}$ are given by $\bar{A}^{\lambda_1}_{\lambda_0}$ and $Z^{\sigma_0}_{\sigma_1}$ respectively. (The most general form of the null vectors of the matrices $\bar{\omega}_{\tau_1\lambda_1}$ and $\hat{\omega}^{\rho_1\sigma_1}$ is $Z^{\lambda_1}_{\alpha_2}\nu^{\alpha_2}$ and $A^{\rho_2}_{\rho_1}\xi_{\rho_2}$, respectively, with ν^{α_2} and ξ_{ρ_2} as arbitrary functions, but this does not affect our proof.) For this reason, the only candidates for null vectors of $\tilde{\omega}_{\tau_1\lambda_1}$ and $\tilde{\omega}^{\rho_1\sigma_1}$ are on the one hand $Z^{\lambda_1}_{\alpha_2}$ and $A^{\rho_2}_{\rho_1}$ respectively and on the other hand $\bar{A}^{\lambda_1}_{\lambda_0}$ and $Z^{\sigma_0}_{\sigma_1}$ respectively. We show that none of these candidates are null vectors. Indeed, from (63) and (64) we find

$$Z^{\lambda_1}_{\alpha_2}\tilde{\omega}_{\tau_1\lambda_1} \approx \bar{D}^{\sigma_2}_{\tau_2}A^{\tau_2}_{\tau_1}\omega_{\sigma_2\alpha_2} \approx \bar{A}^{\sigma_2}_{\tau_1}\omega_{\sigma_2\alpha_2}, \tag{65}$$

$$A^{\rho_2}_{\rho_1}\tilde{\omega}^{\rho_1\sigma_1} \approx D^{\rho_2}_{\alpha_2}\omega^{\alpha_2\beta_2}Z^{\sigma_1}_{\beta_2}. \tag{66}$$

Since $D^{\rho_2}_{\alpha_2}$, $\omega_{\sigma_2\alpha_2}$ and $\omega^{\alpha_2\beta_2}$ are invertible, they have no nontrivial null vectors. On the other hand, the matrix $Z^{\sigma_1}_{\beta_2}\bar{A}^{\sigma_2}_{\sigma_1}$ is of maximum rank (see (34)), so neither $\bar{A}^{\sigma_2}_{\tau_1}$ nor $Z^{\sigma_1}_{\beta_2}$

can display nontrivial null vectors (i.e. there are no nontrivial functions θ_{σ_2} or π^{β_2} such that $\bar{A}^{\sigma_2}_{\tau_1}\theta_{\sigma_2} \approx 0$ or $Z^{\sigma_1}_{\beta_2}\pi^{\beta_2} \approx 0$). In consequence, the objects $Z^{\lambda_1}_{\alpha_2}\tilde{\omega}_{\tau_1\lambda_1}$ and $A^{\rho_2}_{\rho_1}\tilde{\omega}^{\rho_1\sigma_1}$ from (65) and (66) cannot vanish, and therefore the matrices $\tilde{\omega}_{\tau_1\lambda_1}$ and $\tilde{\omega}^{\rho_1\sigma_1}$ do not have the functions $Z^{\lambda_1}_{\alpha_2}$ and $\bar{A}^{\rho_2}_{\rho_1}$ as null vectors respectively. Multiplying (63) and (64) by $\bar{A}^{\lambda_1}_{\lambda_0}$ and $Z^{\sigma_0}_{\sigma_1}$ respectively, we infer the relations

$$\tilde{\omega}_{\tau_1\lambda_1}\bar{A}^{\lambda_1}_{\lambda_0} \approx \bar{\omega}_{\tau_1\lambda_1}\bar{A}^{\lambda_1}_{\lambda_0}, \tag{67}$$

$$\tilde{\omega}^{\rho_1\sigma_1}Z^{\sigma_0}_{\sigma_1} \approx \hat{\omega}^{\rho_1\sigma_1}Z^{\sigma_0}_{\sigma_1}. \tag{68}$$

The right-hand sides of (67) and (68) vanish for

$$\bar{\omega}_{\tau_1\lambda_1} = A^{\sigma_2}_{\tau_1}\bar{\varepsilon}_{\sigma_2\gamma_2}A^{\gamma_2}_{\lambda_1}, \tag{69}$$

$$\hat{\omega}^{\rho_1\sigma_1} = Z^{\rho_1}_{\alpha_2}\hat{\varepsilon}^{\alpha_2\beta_2}Z^{\sigma_1}_{\beta_2}, \tag{70}$$

where $\bar{\varepsilon}_{\sigma_2\gamma_2}$ and $\hat{\varepsilon}^{\alpha_2\beta_2}$ are antisymmetric. It is simple to see that $\bar{\omega}_{\tau_1\lambda_1}$ and $\hat{\omega}^{\rho_1\sigma_1}$ given by (69) and (70) cannot be brought to the form expressed by relations (55) and (57) respectively for any choice of $\bar{\varepsilon}_{\sigma_2\gamma_2}$ or $\hat{\varepsilon}^{\alpha_2\beta_2}$. Thus, it follows that relations (69) and (70) cannot hold, such that $\bar{\omega}_{\tau_1\lambda_1}\bar{A}^{\lambda_1}_{\lambda_0}$ and $\hat{\omega}^{\rho_1\sigma_1}Z^{\sigma_0}_{\sigma_1}$ do not vanish. Therefore, neither $\tilde{\omega}_{\tau_1\lambda_1}$ nor $\tilde{\omega}^{\rho_1\sigma_1}$ (expressed by (63) and (64) respectively) have the functions $\bar{A}^{\lambda_1}_{\lambda_0}$ and $Z^{\sigma_0}_{\sigma_1}$ as null vectors respectively, so they are invertible. This proves (a).

(b) By straightforward computation, it results

$$\tilde{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1} \approx \hat{\omega}^{\rho_1\beta_1}, \tag{71}$$

$$\hat{\omega}^{\rho_1\beta_1}\tilde{\omega}_{\beta_1\lambda_1} \approx \hat{\omega}^{\rho_1\beta_1}\bar{\omega}_{\beta_1\lambda_1} \approx D^{\rho_1}_{\lambda_1}, \tag{72}$$

and hence

$$\tilde{\omega}^{\rho_1\sigma_1}D^{\beta_1}_{\sigma_1}\tilde{\omega}_{\beta_1\lambda_1} \approx D^{\rho_1}_{\lambda_1}, \tag{73}$$

which proves (b).

(c) Taking into account formulae (35), (36) and (58), from relations (63) and (64) we find

$$\tilde{\omega}^{\rho_1\sigma_1}\tilde{\omega}_{\sigma_1\lambda_1} \approx D^{\rho_1}_{\lambda_1} + Z^{\rho_1}_{\alpha_2}\omega^{\alpha_2\beta_2}\omega_{\beta_2\gamma_2}A^{\lambda_2}_{\lambda_1}\bar{D}^{\gamma_2}_{\lambda_2}. \tag{74}$$

Now, we take the matrices $\omega_{\sigma_2\gamma_2}$ and $\omega^{\alpha_2\beta_2}$ to be mutually inverse, namely

$$\omega^{\alpha_2\beta_2}\omega_{\beta_2\gamma_2} \approx \delta^{\alpha_2}_{\gamma_2}. \tag{75}$$

Substituting (75) into (74) and recalling formula (51), we deduce (60). This proves (c).

With these elements at hand, the next theorem is shown to hold. □

Theorem 2. *There exists an invertible, antisymmetric matrix $\mu^{(2)\alpha_0\beta_0}$, in terms of which the Dirac bracket (44) becomes*

$$[F,G]^{(2)*} = [F,G] - \left[F,\chi_{\alpha_0}\right]\mu^{(2)\alpha_0\beta_0}\left[\chi_{\beta_0},G\right] \tag{76}$$

on the surface (1).

Proof. First, we observe that $D^{\alpha_0}_{\gamma_0}$ given in (42) is a projector

$$D^{\alpha_0}_{\gamma_0}D^{\gamma_0}_{\lambda_0} \approx D^{\alpha_0}_{\lambda_0} \tag{77}$$

and satisfies the relations

$$\bar{A}^{\gamma_1}_{\alpha_0}D^{\alpha_0}_{\gamma_0} \approx 0, \qquad D^{\alpha_0}_{\gamma_0}\chi_{\alpha_0} \approx \chi_{\gamma_0}. \tag{78}$$

Multiplying (43) by $\bar{A}^{\gamma_1}_{\alpha_0}$ and using (78), we obtain the equation

$$\bar{A}^{\gamma_1}_{\alpha_0} M^{(2)\alpha_0\beta_0} C^{(2)}_{\beta_0\gamma_0} \approx 0, \tag{79}$$

which then leads to

$$\bar{A}^{\gamma_1}_{\alpha_0} M^{(2)\alpha_0\beta_0} \approx f^{\gamma_1\beta_1} Z^{\beta_0}_{\beta_1}, \tag{80}$$

for some functions $f^{\gamma_1\beta_1}$. Acting with $D^{\tau_0}_{\beta_0}$ on (80) and taking into account (41), we reach the relation

$$\bar{A}^{\gamma_1}_{\alpha_0} M^{(2)\alpha_0\beta_0} D^{\tau_0}_{\beta_0} \approx 0, \tag{81}$$

which combined with the former formula in (78) produces

$$M^{(2)\alpha_0\beta_0} D^{\tau_0}_{\beta_0} \approx \lambda^{\tau_0\beta_0} D^{\alpha_0}_{\beta_0}, \tag{82}$$

for some $\lambda^{\tau_0\beta_0}$. Applying now $D^{\tau_0}_{\alpha_0}$ on (43) and employing relation (82), we deduce

$$-\lambda^{\tau_0\alpha_0} D^{\beta_0}_{\alpha_0} C^{(2)}_{\beta_0\gamma_0} \approx D^{\tau_0}_{\gamma_0}. \tag{83}$$

On the other hand, the latter formula from (78) ensures that

$$D^{\beta_0}_{\alpha_0} C^{(2)}_{\beta_0\gamma_0} \approx C^{(2)}_{\alpha_0\gamma_0}, \tag{84}$$

such that with the aid of the results expressed by (83) and (84) we find

$$-\lambda^{\tau_0\alpha_0} C^{(2)}_{\alpha_0\gamma_0} \approx D^{\tau_0}_{\gamma_0}. \tag{85}$$

Comparing (85) with (43) and recalling that the elements $M^{(2)\alpha_0\beta_0}$ are defined up to transformation (45), we infer the relation

$$M^{(2)\tau_0\alpha_0} = -\lambda^{\tau_0\alpha_0}, \tag{86}$$

which inserted in (82) provides the equation

$$D^{\tau_0}_{\alpha_0} M^{(2)\alpha_0\beta_0} \approx M^{(2)\tau_0\alpha_0} D^{\beta_0}_{\alpha_0}. \tag{87}$$

Using once more the fact that the elements $M^{(2)\alpha_0\beta_0}$ are defined up to (45), from (87) it results

$$M^{(2)\alpha_0\beta_0} \approx D^{\alpha_0}_{\lambda_0} \mu^{(2)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0}, \tag{88}$$

where the elements $\mu^{(2)\lambda_0\sigma_0}$ define an antisymmetric matrix. Based on the former formula from (78) and on relation (88), we infer

$$\bar{A}^{\gamma_1}_{\alpha_0} M^{(2)\alpha_0\beta_0} \approx 0. \tag{89}$$

Replacing (77) in (88), we arrive at

$$D^{\alpha_0}_{\lambda_0} M^{(2)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0} \approx D^{\alpha_0}_{\lambda_0} \mu^{(2)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0}, \tag{90}$$

which leads to

$$\mu^{(2)\lambda_0\sigma_0} \approx M^{(2)\lambda_0\sigma_0} + Z^{\lambda_0}_{\lambda_1} \Omega^{\lambda_1\sigma_1} Z^{\sigma_0}_{\sigma_1}, \tag{91}$$

for some antisymmetric functions $\Omega^{\lambda_1\sigma_1}$. At this point we show that the matrix $\mu^{(2)\lambda_0\sigma_0}$ can indeed be taken to be invertible. If we choose $\Omega^{\lambda_1\sigma_1}$ as $\Omega^{\lambda_1\sigma_1} = \tilde{\omega}^{\lambda_1\sigma_1}$, where $\tilde{\omega}^{\lambda_1\sigma_1}$ is precisely the invertible matrix given in (64), we get

$$\mu^{(2)\lambda_0\sigma_0} \approx M^{(2)\lambda_0\sigma_0} + Z^{\lambda_0}_{\lambda_1} \tilde{\omega}^{\lambda_1\sigma_1} Z^{\sigma_0}_{\sigma_1}. \tag{92}$$

In the following, we show that the matrix of elements

$$\mu^{(2)}_{\sigma_0\rho_0} \approx C^{(2)}_{\sigma_0\rho_0} + \bar{A}^{\rho_1}_{\sigma_0} \tilde{\omega}_{\rho_1\tau_1} \bar{A}^{\tau_1}_{\rho_0}, \tag{93}$$

with $\tilde{\omega}_{\rho_1\tau_1}$ the invertible matrix from (63), is nothing but the inverse of $\mu^{(2)\lambda_0\sigma_0}$ expressed in (92). Indeed, relying on relations (33), (40), (43) and (89), by direct computation we find

$$\mu^{(2)\lambda_0\sigma_0}\mu^{(2)}_{\sigma_0\rho_0} \approx D^{\lambda_0}_{\rho_0} + Z^{\lambda_0}_{\lambda_1}\tilde{\omega}^{\lambda_1\sigma_1}D^{\rho_1}_{\sigma_1}\tilde{\omega}_{\rho_1\tau_1}\bar{A}^{\tau_1}_{\rho_0}. \tag{94}$$

Employing theorem 1 (see (59)) and the former formula in (39), we deduce the relation

$$Z^{\lambda_0}_{\lambda_1}\tilde{\omega}^{\lambda_1\sigma_1}D^{\rho_1}_{\sigma_1}\tilde{\omega}_{\rho_1\tau_1}\bar{A}^{\tau_1}_{\rho_0} \approx Z^{\lambda_0}_{\lambda_1}D^{\lambda_1}_{\tau_1}\bar{A}^{\tau_1}_{\rho_0} \approx Z^{\lambda_0}_{\lambda_1}\bar{A}^{\lambda_1}_{\rho_0}, \tag{95}$$

which replaced in (94) reduces to

$$\mu^{(2)\lambda_0\sigma_0}\mu^{(2)}_{\sigma_0\rho_0} \approx \delta^{\lambda_0}_{\rho_0}. \tag{96}$$

The above formula proves that the matrix of elements $\mu^{(2)\lambda_0\sigma_0}$ from (92) is (weakly) invertible and therefore completes the proof of this theorem. □

Formula (76) plays a key role in what follows. It allows one to express the original Dirac bracket (31), initially written only in terms of a *subset of independent* second-class constraint functions, with the help of an invertible matrix, whose indices cover the *whole set* of reducible second-class constraints. Inspired by this result, we will be able to find an irreducible second-class constraint set, whose Dirac bracket is (weakly) equal to (76).

3.2. Irreducible approach

3.2.1. Intermediate system. Now, we introduce some new variables, $(y_{\alpha_1})_{\alpha_1=1,\ldots,M_1}$, independent of the original phase-space variables z^a, with the Poisson brackets

$$[y_{\alpha_1}, y_{\beta_1}] = \omega_{\alpha_1\beta_1}, \tag{97}$$

where the elements $\omega_{\alpha_1\beta_1}$ define an invertible, antisymmetric (but otherwise arbitrary) matrix, and consider the system subject to the reducible second-class constraints

$$\chi_{\alpha_0} \approx 0, \qquad y_{\alpha_1} \approx 0. \tag{98}$$

(The elements $\omega_{\alpha_1\beta_1}$ may depend at most on the newly added variables, just like the objects $\Gamma_{\alpha_1\beta_1}$ from section 2.2.) The system subject to the second-class constraints (98) will be called an *intermediate system* in what follows. The Dirac bracket on the larger phase space, locally described by (z^a, y_{α_1}), corresponding to the above second-class constraints reads as

$$[F,G]^{(2)*}|_{z,y} = [F,G] - [F,\chi_{\alpha_0}]\mu^{(2)\alpha_0\beta_0}[\chi_{\beta_0},G] - [F,y_{\alpha_1}]\omega^{\alpha_1\beta_1}[y_{\beta_1},G], \tag{99}$$

where the Poisson brackets from the right-hand side of (99) contain derivatives with respect to all z^a's and y_{α_1}'s, and $\omega^{\alpha_1\beta_1}$ denotes the elements of the inverse of $\omega_{\alpha_1\beta_1}$. On the one hand, the most general form of a smooth function defined on the phase space with the local coordinates (z^a, y_{α_1}) is

$$F(z^a, y_{\alpha_1}) = F_0(z^a) + b^{\lambda_1}(z^a)y_{\lambda_1} + b^{\lambda_1\rho_1}(z^a)y_{\lambda_1}y_{\rho_1} + \cdots, \tag{100}$$

for some smooth functions $b^{\lambda_1}(z^a)$, $b^{\lambda_1\rho_1}(z^a)$, etc. On the other hand, direct computation yields

$$[F,G]^{(2)*} \approx [F_0,G_0]^{(2)*}, \tag{101}$$

where the previous weak equality is defined on the surface (98). Moreover, equations (1) and (98) describe the same surface, but embedded in phase spaces of different dimensions. In other words, equations (1) and (98) are equivalent descriptions of the *same surface of constraints*. For this reason, we will employ the same symbol of weak equality for both descriptions. (It is understood that if we work with functions defined on the phase space of coordinates z^a, then we employ representation (1), but if we work with functions of $\left(z^a, y_{\alpha_1}\right)$, then we use (98).) Inserting (100) in (99) and taking (101) into account, we obtain

$$[F, G]^{(2)*}|_{z,y} \approx [F, G]^{(2)*}. \tag{102}$$

We recall that the Dirac bracket $[F, G]^{(2)*}$ contains only derivatives with respect to the original variables z^a.

Formula (102) is important since together with (76) it opens the perspective toward the construction of an irreducible second-class constraint system associated with the original, second-order reducible one, but on the larger phase space $\left(z^a, y_{\alpha_1}\right)$.

3.2.2. Irreducible system. Now, we choose $\omega_{\gamma_1\lambda_1}$ from (97) such that

$$\tilde{\omega}_{\alpha_1\beta_1} = \hat{E}^{\gamma_1}_{\alpha_1}\omega_{\gamma_1\lambda_1}\hat{E}^{\lambda_1}_{\beta_1}, \tag{103}$$

for an invertible matrix, of elements $\hat{E}^{\gamma_1}_{\alpha_1}$, with the help of which we introduce the functions

$$A^{\rho_1}_{\sigma_0} = \hat{E}^{\rho_1}_{\alpha_1}\bar{A}^{\alpha_1}_{\sigma_0}. \tag{104}$$

Then, we have that

$$\tilde{\omega}^{\alpha_1\beta_1} = \hat{e}^{\alpha_1}_{\sigma_1}\omega^{\sigma_1\tau_1}\hat{e}^{\beta_1}_{\tau_1}, \tag{105}$$

where $\hat{e}^{\alpha_1}_{\sigma_1}$ is the inverse of $\hat{E}^{\gamma_1}_{\alpha_1}$. By means of (104), we find

$$\bar{A}^{\alpha_1}_{\sigma_0} = A^{\rho_1}_{\sigma_0}\hat{e}^{\alpha_1}_{\rho_1}. \tag{106}$$

In this context, the following theorem can be shown to hold.

Theorem 3. *The elements $\hat{e}^{\alpha_1}_{\sigma_1}$ and $\hat{E}^{\tau_1}_{\beta_1}$ can always be taken such that*

$$\hat{E}^{\alpha_1}_{\sigma_1}D^{\sigma_1}_{\tau_1}\hat{e}^{\tau_1}_{\beta_1} \approx D^{\alpha_1}_{\beta_1}. \tag{107}$$

Proof. We choose $\hat{E}^{\alpha_1}_{\beta_1}$ such that

$$A^{\alpha_1}_{\alpha_0} = \sigma_{\alpha_0\beta_0}\sigma^{\alpha_1\beta_1}Z^{\beta_0}_{\beta_1}, \tag{108}$$

where $\sigma_{\alpha_0\beta_0}$ is invertible and $\sigma^{\alpha_1\beta_1}$ is invertible and symmetric. If we take

$$A^{\alpha_2}_{\alpha_1} = \sigma_{\alpha_1\lambda_1}\sigma^{\alpha_2\beta_2}Z^{\lambda_1}_{\beta_2}, \tag{109}$$

with $\sigma^{\alpha_2\beta_2}$ invertible and $\sigma_{\alpha_1\lambda_1}$ the inverse of $\sigma^{\alpha_1\beta_1}$, then we obtain that (50) is satisfied. (With this choice of $A^{\alpha_2}_{\alpha_1}$, we have that $D^{\alpha_2}_{\lambda_2} = Z^{\alpha_1}_{\lambda_2}\sigma_{\alpha_1\lambda_1}Z^{\lambda_1}_{\beta_2}\sigma^{\alpha_2\beta_2}$. Because $Z^{\alpha_1}_{\lambda_2}$ has no nontrivial null vectors, it follows that the matrix of elements $Z^{\alpha_1}_{\lambda_2}\sigma_{\alpha_1\lambda_1}Z^{\lambda_1}_{\beta_2}$ is invertible. On the other hand, $\sigma^{\alpha_2\beta_2}$ is by hypothesis invertible, so $D^{\alpha_2}_{\lambda_2}$ is the same, as required by (50).) Employing (108) and (109) and recalling (30), we get

$$A^{\alpha_2}_{\alpha_1}A^{\alpha_1}_{\alpha_0} \approx 0. \tag{110}$$

Expressing the first-order reducibility functions from (108) and (109)

$$Z^{\alpha_0}_{\alpha_1} = \sigma^{\alpha_0\beta_0}\sigma_{\alpha_1\beta_1}A^{\beta_1}_{\beta_0}, \qquad Z^{\lambda_1}_{\lambda_2} = \sigma^{\lambda_1\tau_1}\sigma_{\lambda_2\tau_2}A^{\tau_2}_{\tau_1}, \tag{111}$$

where $\sigma^{\alpha_0\beta_0}$ and $\sigma_{\lambda_2\tau_2}$ are the inverses of $\sigma_{\alpha_0\beta_0}$ and $\sigma^{\alpha_2\beta_2}$ respectively, we deduce

$$Z^{\alpha_0}_{\alpha_1}\hat{e}^{\alpha_1}_{\lambda_1}Z^{\lambda_1}_{\lambda_2} = \sigma^{\alpha_0\beta_0}\sigma_{\lambda_2\tau_2}A^{\beta_1}_{\beta_0}\sigma_{\alpha_1\beta_1}\hat{e}^{\alpha_1}_{\lambda_1}\sigma^{\lambda_1\tau_1}A^{\tau_2}_{\tau_1}. \tag{112}$$

Formula (105) can be rewritten as $\tilde{\omega}^{\alpha_1\beta_1} = \hat{e}^{\alpha_1\sigma_1}\check{\omega}_{\sigma_1\tau_1}\hat{e}^{\beta_1\tau_1}$, with $\check{\omega}_{\sigma_1\tau_1} = \sigma_{\sigma_1\rho_1}\omega^{\rho_1\gamma_1}\sigma_{\gamma_1\tau_1}$ and $\hat{e}^{\alpha_1\sigma_1} = \hat{e}^{\alpha_1}_{\lambda_1}\sigma^{\lambda_1\sigma_1}$. Because the matrix $\sigma_{\sigma_1\rho_1}$ is symmetric and $\omega^{\rho_1\gamma_1}$ antisymmetric, it follows that $\check{\omega}_{\sigma_1\tau_1}$ is antisymmetric. The antisymmetry property of both $\tilde{\omega}^{\alpha_1\beta_1}$ and $\check{\omega}_{\sigma_1\tau_1}$ implies that the quantities $\hat{e}^{\alpha_1\sigma_1}$ can be taken to be symmetric

$$\hat{e}^{\alpha_1\sigma_1} = \hat{e}^{\alpha_1}_{\lambda_1}\sigma^{\lambda_1\sigma_1} = \hat{e}^{\sigma_1\alpha_1}. \tag{113}$$

(The other possibility, namely the antisymmetry of $\hat{e}^{\alpha_1\sigma_1}$, will not be considered in the following.) By means of (113) we infer $\sigma_{\alpha_1\beta_1}\hat{e}^{\alpha_1}_{\lambda_1}\sigma^{\lambda_1\tau_1} = \hat{e}^{\tau_1}_{\beta_1}$, such that from (112) (and also (106)) we find the relation

$$Z^{\alpha_0}_{\alpha_1}\hat{e}^{\alpha_1}_{\lambda_1}Z^{\lambda_1}_{\lambda_2} = \sigma^{\alpha_0\beta_0}\sigma_{\lambda_2\tau_2}\bar{A}^{\beta_1}_{\beta_0}A^{\tau_2}_{\beta_1}. \tag{114}$$

Substituting now (53) in (114), we obtain

$$Z^{\alpha_0}_{\alpha_1}\hat{e}^{\alpha_1}_{\lambda_1}Z^{\lambda_1}_{\lambda_2} \approx 0. \tag{115}$$

With relations (110) and (115) at hand, we are in a position to prove (107). If we make the notation

$$\hat{D}^{\alpha_1}_{\beta_1} = \hat{e}^{\alpha_1}_{\sigma_1}D^{\sigma_1}_{\tau_1}\hat{E}^{\tau_1}_{\beta_1}, \tag{116}$$

then it is easy to see that $\hat{D}^{\alpha_1}_{\beta_1}$ is a projector

$$\hat{D}^{\alpha_1}_{\beta_1}\hat{D}^{\beta_1}_{\lambda_1} \approx \hat{D}^{\alpha_1}_{\lambda_1}. \tag{117}$$

On the other hand, with the aid of (104) and (110) we deduce

$$\bar{A}^{\beta_1}_{\alpha_0}\hat{D}^{\alpha_1}_{\beta_1} \approx \bar{A}^{\alpha_1}_{\alpha_0}. \tag{118}$$

Applying $Z^{\alpha_0}_{\alpha_1}$ on (116) and using (115), it follows

$$Z^{\alpha_0}_{\alpha_1}\hat{D}^{\alpha_1}_{\beta_1} \approx Z^{\alpha_0}_{\beta_1}. \tag{119}$$

Multiplying (118) with $Z^{\alpha_0}_{\rho_1}$ and (119) with $\bar{A}^{\alpha_1}_{\alpha_0}$, we reach the equations

$$\hat{D}^{\alpha_1}_{\beta_1}D^{\beta_1}_{\rho_1} \approx D^{\alpha_1}_{\rho_1}, \qquad D^{\alpha_1}_{\beta_1}\hat{D}^{\beta_1}_{\rho_1} \approx D^{\alpha_1}_{\rho_1}. \tag{120}$$

The general solution to equations (120) can be represented like

$$\hat{D}^{\alpha_1}_{\beta_1} \approx D^{\alpha_1}_{\beta_1} + Z^{\alpha_1}_{\lambda_2}M^{\lambda_2}_{\tau_2}A^{\tau_2}_{\beta_1}, \tag{121}$$

for some matrix $M^{\lambda_2}_{\tau_2}$. Direct computation shows that

$$\hat{D}^{\alpha_1}_{\beta_1}\hat{D}^{\beta_1}_{\lambda_1} \approx D^{\alpha_1}_{\lambda_1} + Z^{\alpha_1}_{\lambda_2}M^{\lambda_2}_{\tau_2}D^{\tau_2}_{\beta_2}M^{\beta_2}_{\rho_2}A^{\rho_2}_{\lambda_1}. \tag{122}$$

Comparing (122) with (117) and employing (121), we find that $M^{\lambda_2}_{\tau_2}$ are solutions to the equations

$$Z^{\alpha_1}_{\lambda_2}M^{\lambda_2}_{\tau_2}D^{\tau_2}_{\beta_2}M^{\beta_2}_{\rho_2}A^{\rho_2}_{\lambda_1} \approx Z^{\alpha_1}_{\lambda_2}M^{\lambda_2}_{\tau_2}A^{\tau_2}_{\lambda_1}. \tag{123}$$

It is simple to see that equation (123) possesses two kinds of solutions, namely

$$M^{\lambda_2}_{\tau_2} = 0 \tag{124}$$

and

$$M^{\lambda_2}_{\tau_2} = \bar{D}^{\lambda_2}_{\tau_2} \tag{125}$$

respectively. If we take the second solution, (124), from (121), we obtain

$$\hat{D}^{\alpha_1}_{\beta_1} \approx D^{\alpha_1}_{\beta_1}, \tag{126}$$

which ensures (107). (Solution (125) leads to the equation $\hat{e}^{\alpha_1}_{\sigma_1}D^{\sigma_1}_{\tau_1}\hat{E}^{\tau_1}_{\beta_1} \approx \delta^{\alpha_1}_{\beta_1}$. This further provides the relation $D^{\sigma_1}_{\beta_1} \approx \delta^{\alpha_1}_{\beta_1}$, which contradicts (51).) This proves the theorem. □

Inserting (103)–(105) in (59) and recalling (107), it is easy to deduce the relation

$$\omega^{\alpha_1\tau_1} D^{\sigma_1}_{\tau_1} \omega_{\sigma_1\beta_1} \approx D^{\alpha_1}_{\beta_1}. \tag{127}$$

On the other hand, formulae (103)–(105) indicate that $\mu^{(2)\lambda_0\sigma_0}$ and $\mu^{(2)}_{\sigma_0\rho_0}$ provided by (92) and (93) take the form

$$\mu^{(2)\lambda_0\sigma_0} \approx M^{(2)\lambda_0\sigma_0} + Z^{\lambda_0}_{\lambda_1} \hat{e}^{\lambda_1}_{\sigma_1} \omega^{\sigma_1\tau_1} \hat{e}^{\gamma_1}_{\tau_1} Z^{\sigma_0}_{\gamma_1}, \tag{128}$$

$$\mu^{(2)}_{\sigma_0\rho_0} \approx C^{(2)}_{\sigma_0\rho_0} + A^{\rho_1}_{\sigma_0} \omega_{\rho_1\tau_1} A^{\tau_1}_{\rho_0}. \tag{129}$$

At this point, we have all the necessary ingredients (objects and their properties) for unfolding the irreducible approach. We introduce the constraints

$$\tilde{\chi}_{\alpha_0} = \chi_{\alpha_0} + A^{\alpha_1}_{\alpha_0} y_{\alpha_1} \approx 0, \qquad \tilde{\chi}_{\alpha_2} = Z^{\alpha_1}_{\alpha_2} y_{\alpha_1} \approx 0, \tag{130}$$

defined on the larger phase space $\left(z^{\Delta}, y_{\alpha_1}\right)$. In the following, we show that (130) *display all the desired properties*: equivalence with the intermediate system (98), second-class behavior, irreducibility and, most important, the associated Dirac bracket coincides (weakly) with the original one, corresponding to the second-order reducible second-class constraints. The proof of all these properties is contained within the next two theorems.

Theorem 4. *Constraints (130) exhibit the following properties:*

(i) equivalence to (98), i.e.

$$\left(\tilde{\chi}_{\alpha_0} \approx 0, \tilde{\chi}_{\alpha_2} \approx 0\right) \quad \Leftrightarrow \quad \left(\chi_{\alpha_0} \approx 0, y_{\alpha_1} \approx 0\right); \tag{131}$$

(ii) second-class behavior, i.e. the matrix

$$C_{\Delta\Delta'} = [\tilde{\chi}_{\Delta}, \tilde{\chi}_{\Delta'}] \tag{132}$$

is invertible, where

$$\tilde{\chi}_{\Delta} = \left(\tilde{\chi}_{\alpha_0}, \tilde{\chi}_{\alpha_2}\right); \tag{133}$$

(iii) irreducibility.

Proof. Due to the equivalence (131), in what follows we will use the same symbol of weak equality in relation with each constraint set (98) and (130).

(i) It is easy to see that if (98) holds, then (130) also holds:

$$\left(\chi_{\alpha_0} \approx 0, y_{\alpha_1} \approx 0\right) \quad \Rightarrow \quad \left(\tilde{\chi}_{\alpha_0} \approx 0, \tilde{\chi}_{\alpha_2} \approx 0\right). \tag{134}$$

By means of relations (104) and (107), from (130) we infer

$$\chi_{\alpha_0} = D^{\beta_0}_{\alpha_0} \tilde{\chi}_{\beta_0}, \qquad y_{\alpha_1} = Z^{\alpha_0}_{\gamma_1} \hat{e}^{\gamma_1}_{\alpha_1} \tilde{\chi}_{\alpha_0} + A^{\beta_2}_{\alpha_1} \bar{D}^{\alpha_2}_{\beta_2} \tilde{\chi}_{\alpha_2}. \tag{135}$$

From (135) we obtain that if (130) is satisfied, then (98) is also valid

$$\left(\tilde{\chi}_{\alpha_0} \approx 0, \tilde{\chi}_{\alpha_2} \approx 0\right) \quad \Rightarrow \quad \left(\chi_{\alpha_0} \approx 0, y_{\alpha_1} \approx 0\right). \tag{136}$$

Relations (134) and (136) prove (i).

(ii) By means of (130) and (135), we find the Poisson brackets among the functions $\tilde{\chi}_{\Delta}$ in the form

$$\left[\tilde{\chi}_{\alpha_0}, \tilde{\chi}_{\beta_0}\right] \approx \mu^{(2)}_{\alpha_0\beta_0}, \qquad \left[\tilde{\chi}_{\alpha_0}, \tilde{\chi}_{\beta_2}\right] \approx A^{\alpha_1}_{\alpha_0} \omega_{\alpha_1\beta_1} Z^{\beta_1}_{\beta_2}, \tag{137}$$

$$\left[\tilde{\chi}_{\alpha_2}, \tilde{\chi}_{\beta_2}\right] \approx Z^{\alpha_1}_{\alpha_2} \omega_{\alpha_1\beta_1} Z^{\beta_1}_{\beta_2}, \tag{138}$$

where $\mu^{(2)}_{\alpha_0\beta_0}$ is given by (129). Then, the matrix $C_{\Delta\Delta'}$ takes the concrete form

$$C_{\Delta\Delta'} = \begin{pmatrix} \mu^{(2)}_{\alpha_0\beta_0} & A^{\alpha_1}_{\alpha_0}\omega_{\alpha_1\beta_1}Z^{\beta_1}_{\beta_2} \\ Z^{\alpha_1}_{\alpha_2}\omega_{\alpha_1\beta_1}A^{\beta_1}_{\beta_0} & Z^{\alpha_1}_{\alpha_2}\omega_{\alpha_1\beta_1}Z^{\beta_1}_{\beta_2} \end{pmatrix}, \tag{139}$$

where $\Delta = (\alpha_0, \alpha_2)$ indexes the line and $\Delta' = (\beta_0, \beta_2)$ the column. In order to prove that $C_{\Delta\Delta'}$ is invertible, we will simply exhibit its inverse. Direct computation based on relations (107), (110), (115), (127) and (128) shows that

$$C^{\Delta'\Delta''} = \begin{pmatrix} \mu^{(2)\beta_0\rho_0} & Z^{\beta_0}_{\gamma_1}\hat{e}^{\gamma_1}_{\sigma_1}\omega^{\sigma_1\lambda_1}A^{\tau_2}_{\lambda_1}\bar{D}^{\rho_2}_{\tau_2} \\ \bar{D}^{\beta_2}_{\lambda_2}A^{\lambda_2}_{\sigma_1}\omega^{\sigma_1\lambda_1}\hat{e}^{\gamma_1}_{\lambda_1}Z^{\rho_0}_{\gamma_1} & \bar{D}^{\beta_2}_{\lambda_2}A^{\lambda_2}_{\sigma_1}\omega^{\sigma_1\lambda_1}A^{\tau_2}_{\lambda_1}\bar{D}^{\rho_2}_{\tau_2} \end{pmatrix}, \tag{140}$$

with $\mu^{(2)\beta_0\rho_0}$ as in (129) satisfies the relations

$$C_{\Delta\Delta'}C^{\Delta'\Delta''} \approx \begin{pmatrix} \delta^{\rho_0}_{\alpha_0} & 0 \\ 0 & \delta^{\rho_2}_{\alpha_2} \end{pmatrix}, \tag{141}$$

and hence the matrix of elements (139) is invertible, its inverse being precisely (140). This proves (ii).

(iii) As the matrix (139) is invertible, it possesses no nontrivial null vectors. In consequence, the functions $\tilde{\chi}_\Delta$ are all independent, so the constraint set (130) is indeed irreducible. This proves (iii). □

By means of result (140), the Dirac bracket associated with the irreducible second-class constraints (130)

$$[F, G]^{(2)*}|_{\text{ired}} = [F, G] - [F, \tilde{\chi}_\Delta]C^{\Delta\Delta'}[\tilde{\chi}_{\Delta'}, G] \tag{142}$$

takes the concrete form

$$\begin{aligned}[F, G]^{(2)*}|_{\text{ired}} = {} & [F, G] - [F, \tilde{\chi}_{\alpha_0}]\mu^{(2)\alpha_0\beta_0}[\tilde{\chi}_{\beta_0}, G] - [F, \tilde{\chi}_{\alpha_0}]Z^{\alpha_0}_{\gamma_1}\hat{e}^{\gamma_1}_{\sigma_1}\omega^{\sigma_1\lambda_1}A^{\tau_2}_{\lambda_1}\bar{D}^{\beta_2}_{\tau_2}[\tilde{\chi}_{\beta_2}, G] \\ & - [F, \tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\sigma_1}\omega^{\sigma_1\lambda_1}\hat{e}^{\gamma_1}_{\lambda_1}Z^{\beta_0}_{\gamma_1}[\tilde{\chi}_{\beta_0}, G] \\ & - [F, \tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\sigma_1}\omega^{\sigma_1\lambda_1}A^{\tau_2}_{\lambda_1}\bar{D}^{\beta_2}_{\tau_2}[\tilde{\chi}_{\beta_2}, G].\end{aligned} \tag{143}$$

We observe that the first line from the right-hand side of (143) is generated by the first-order reducibility relations (see (27)), while the remaining terms are due to the second-order reducibility functions. Together with (130), *formula* (143) *is the corner stone of our irreducible approach*. We will show that it coincides (weakly) with the Dirac bracket of the intermediate system, and therefore with the original Dirac bracket for the second-order reducible second-class constraints.

Theorem 5. *The Dirac bracket with respect to the irreducible second-class constraints, (143), coincides with that of the intermediate system*

$$[F, G]^{(2)*}|_{\text{ired}} \approx [F, G]^{(2)*}|_{z,y}. \tag{144}$$

Proof. In order to prove the theorem we start from the right-hand side of (143) and show that it is weakly equal to the right-hand side of (99). Using relations (104), (107), (128) and (129), by direct computation we find that

$$[F, \tilde{\chi}_{\alpha_0}]\mu^{(2)\alpha_0\beta_0}[\tilde{\chi}_{\beta_0}, G] \approx [F, \chi_{\alpha_0}]\mu^{(2)\alpha_0\beta_0}[\chi_{\beta_0}, G] + [F, y_{\alpha_1}]D^{\alpha_1}_{\sigma_1}\omega^{\sigma_1\lambda_1}D^{\beta_1}_{\lambda_1}[y_{\beta_1}, G], \tag{145}$$

$$[F, \tilde{\chi}_{\alpha_0}]Z^{\alpha_0}_{\gamma_1}\hat{e}^{\gamma_1}_{\sigma_1}\omega^{\sigma_1\lambda_1}A^{\tau_2}_{\lambda_1}\bar{D}^{\beta_2}_{\tau_2}[\tilde{\chi}_{\beta_2}, G] \approx [F, y_{\alpha_1}]D^{\alpha_1}_{\sigma_1}\omega^{\sigma_1\lambda_1}\left(\delta^{\beta_1}_{\lambda_1} - D^{\beta_1}_{\lambda_1}\right)[y_{\beta_1}, G], \tag{146}$$

$$[F, \tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\sigma_1}\omega^{\sigma_1\lambda_1}\hat{e}^{\gamma_1}_{\lambda_1}Z^{\beta_0}_{\gamma_1}[\tilde{\chi}_{\beta_0}, G] \approx [F, y_{\alpha_1}]\left(\delta^{\alpha_1}_{\sigma_1} - D^{\alpha_1}_{\sigma_1}\right)\omega^{\sigma_1\lambda_1}D^{\beta_1}_{\lambda_1}[y_{\beta_1}, G], \tag{147}$$

$$\left[F, \tilde{\chi}_{\alpha_2}\right]\bar{D}^{\alpha_2}_{\lambda_2}A^{\lambda_2}_{\sigma_1}\omega^{\sigma_1\lambda_1}A^{\tau_2}_{\lambda_1}\bar{D}^{\beta_2}_{\tau_2}\left[\tilde{\chi}_{\beta_2}, G\right] \approx \left[F, y_{\alpha_1}\right]\left(\delta^{\alpha_1}_{\sigma_1} - D^{\alpha_1}_{\sigma_1}\right)\omega^{\sigma_1\lambda_1}\left(\delta^{\beta_1}_{\lambda_1} - D^{\beta_1}_{\lambda_1}\right)\left[y_{\beta_1}, G\right]. \tag{148}$$

Inserting the above relations into (143), we find (144). This proves the theorem. □

3.3. Main result

Combining (102) and (144), we reach the result

$$[F, G]^{(2)*} \approx [F, G]^{(2)*}|_{\text{ired}}. \tag{149}$$

The last formula proves that we can approach second-order reducible second-class constraints in an irreducible fashion. Thus, starting with the second-order reducible constraints (1) we construct the irreducible constraints (130), whose Poisson brackets form an invertible matrix. Formula (149) ensures that the Dirac bracket within the irreducible setting coincides with that from the reducible version. This is the main result of the present paper.

Moreover, the new variables, y_{α_1}, do not affect the irreducible Dirac bracket as from (143) we have that $\left[y_{\alpha_1}, F\right]^{(2)*}|_{\text{ired}} \approx 0$. Thus, the equations of motion for the original reducible system can be written as $\dot{z}^a \approx [z^a, H]^{(2)*}|_{\text{ired}}$, where H is the canonical Hamiltonian. The equations of motion for y_{α_1} read as $\dot{y}_{\alpha_1} \approx 0$, and lead to $y_{\alpha_1} = 0$ by taking some appropriate boundary conditions (vacuum to vacuum) for these unphysical variables. This completes the general procedure.

4. Example

We exemplify the general results exposed in the above in the case of a field theory—gauge-fixed 3-forms, subject to the second-class constraints

$$\chi_{\alpha_0} \equiv \begin{pmatrix} -3\partial^{i_3}\pi_{i_3i_1i_2} \\ -\partial_{j_3}A^{j_3j_1j_2} \end{pmatrix} \approx 0. \tag{150}$$

Thus, the constraints (150) are second-stage reducible, the first- and the second-stage reducibility matrices being respectively given by

$$Z^{\alpha_0}_{\alpha_1} = \begin{pmatrix} Z^{i_1i_2}_{k_1} & \mathbf{0} \\ \mathbf{0} & Z^{l_1}_{j_1j_2} \end{pmatrix}, \qquad Z^{\alpha_1}_{\alpha_2} = \begin{pmatrix} Z^{k_1} & \mathbf{0} \\ \mathbf{0} & Z_{l_1} \end{pmatrix}, \tag{151}$$

with

$$Z^{i_1i_2}_{k_1} = \delta^{[i_1}_{k_1}\partial^{i_2]}, \qquad Z^{l_1}_{j_1j_2} = \delta^{l_1}_{[j_1}\partial_{j_2]}, \qquad Z^{k_1} = \partial^{k_1}, \qquad Z_{l_1} = \partial_{l_1}. \tag{152}$$

The matrix of the Poisson brackets among the constraints (150) is expressed by

$$C_{\alpha_0\beta_0} = \begin{pmatrix} \mathbf{0} & \Delta D^{i_3i_4}_{i_1i_2} \\ -\Delta D^{j_1j_2}_{j_3j_4} & \mathbf{0} \end{pmatrix}, \tag{153}$$

where

$$D^{i_3i_4}_{i_1i_2} = \frac{1}{2}\left(\delta^{i_3}_{[i_1}\delta^{i_4}_{i_2]} - \frac{\delta^{[i_4}_{k}\partial^{i_3]}\delta^{k}_{[i_2}\partial_{i_1]}}{\Delta}\right) \tag{154}$$

and $\Delta = \partial^i\partial_i$. If we take

$$A^{\beta_2}_{\alpha_1} = \begin{pmatrix} Z_{k_1} & \mathbf{0} \\ \mathbf{0} & Z^{l_1} \end{pmatrix}, \tag{155}$$

then we obtain

$$D^{\beta_2}_{\alpha_2} = Z^{\alpha_1}_{\alpha_2} A^{\beta_2}_{\alpha_1} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \tag{156}$$

such that

$$\bar{D}^{\alpha_2}_{\lambda_2} = \begin{pmatrix} \frac{1}{\Delta} & 0 \\ 0 & \frac{1}{\Delta} \end{pmatrix}. \tag{157}$$

We remark that $A^{\beta_2}_{\beta_1}$ given by (155) can be expressed like in (109) for

$$\sigma_{\alpha_1\beta_1} = \begin{pmatrix} \mathbf{0} & \delta^{k_2}_{k_1} \\ \delta^{l_1}_{l_2} & \mathbf{0} \end{pmatrix} \tag{158}$$

and

$$\sigma^{\alpha_2\beta_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{159}$$

With the help of (151) and (155)–(157), from (51) we find that

$$D^{\alpha_1}_{\beta_1} = \begin{pmatrix} D^{k_1}_{k_2} & \mathbf{0} \\ \mathbf{0} & D^{l_2}_{l_1} \end{pmatrix}, \tag{160}$$

where

$$D^i_j = \delta^i_j - \frac{\partial^i \partial_j}{\Delta}. \tag{161}$$

On the other hand, we can set $D^{\alpha_1}_{\beta_1}$ in the form expressed by (40) by choosing

$$\bar{A}^{\alpha_1}_{\beta_0} = \begin{pmatrix} \frac{1}{2\Delta} Z^{k_1}_{i_3 i_4} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2\Delta} Z^{j_3 j_4}_{l_1} \end{pmatrix}. \tag{162}$$

Then, it is easy to see that

$$Z^{\alpha_0}_{\alpha_1} \bar{A}^{\alpha_1}_{\beta_0} = \begin{pmatrix} \frac{1}{2\Delta} \delta^{[i_2}_{k_1} \partial^{i_1]} \delta^{k_1}_{[i_4} \partial_{i_3]} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2\Delta} \delta^{[j_4}_{l_1} \partial^{j_3]} \delta^{l_1}_{[j_2} \partial_{j_1]} \end{pmatrix}, \tag{163}$$

such that with the aid of (42) we find

$$D^{\alpha_0}_{\beta_0} = \begin{pmatrix} D^{i_1 i_2}_{i_3 i_4} & \mathbf{0} \\ \mathbf{0} & D^{j_3 j_4}_{j_1 j_2} \end{pmatrix}. \tag{164}$$

Based on the fact that $D^{i_1 i_2}_{i_3 i_4}$ is a projector, i.e.

$$D^{i_1 i_2}_{i_3 i_4} D^{i_3 i_4}_{j_1 j_2} = D^{i_1 i_2}_{j_1 j_2}, \tag{165}$$

from (43) and (153) we obtain that

$$M^{(2)\alpha_0\beta_0} = \begin{pmatrix} \mathbf{0} & -\frac{1}{\Delta} D^{i_1 i_2}_{i_3 i_4} \\ \frac{1}{\Delta} D^{j_3 j_4}_{j_1 j_2} & \mathbf{0} \end{pmatrix}. \tag{166}$$

With the help of (44) and (166), we have that the fundamental Dirac brackets read as

$$[A^{ijk}(x), \pi_{i'j'k'}(y)]^{(2)*}_{x^0=y^0} = D^{ijk}_{i'j'k'} \delta^{D-1}(\mathbf{x}-\mathbf{y}), \tag{167}$$

$$[A^{ijk}(x), A^{i'j'k'}(y)]^{(2)*}_{x^0=y^0} = 0, \qquad [\pi_{ijk}(x), \pi_{i'j'k'}(y)]^{(2)*}_{x^0=y^0} = 0, \tag{168}$$

where $D^{ijk}_{i'j'k'}$ is also a projector, expressed by

$$D^{ijk}_{i'j'k'} = \frac{1}{3!}\left(\delta^{i}_{[i'}\delta^{j}_{j'}\delta^{k}_{k']} - \frac{\partial^{[i}\delta^{j}_{l_1}\delta^{k]}_{l_2}\partial_{[i'}\delta^{l_1}_{j'}\delta^{l_2}_{k']}}{2\Delta}\right). \tag{169}$$

Formula (88) together with (164) and (166) provides

$$\mu^{(2)\alpha_0\beta_0} = \begin{pmatrix} \mathbf{0} & -\frac{1}{2\Delta}\delta^{i_1}_{[i_3}\delta^{i_2}_{i_4]} \\ \frac{1}{2\Delta}\delta^{j_3}_{[j_1}\delta^{j_4}_{j_2]} & \mathbf{0} \end{pmatrix}, \tag{170}$$

which clearly exhibits that $\mu^{(2)\alpha_0\beta_0}$ is invertible. By computing the fundamental Dirac brackets with the help of (76) (with $\mu^{(2)\alpha_0\beta_0}$ given by (170)), we reobtain precisely (167) and (168).

On the other hand, using the former relation in (151) as well as (166) and (170) into (92) produces

$$\tilde{\omega}^{\gamma_1\rho_1} = \begin{pmatrix} \mathbf{0} & \frac{1}{2\Delta^2}\delta^{m_1}_{m_2} \\ -\frac{1}{2\Delta^2}\delta^{n_2}_{n_1} & \mathbf{0} \end{pmatrix}. \tag{171}$$

Simple computation shows that $\tilde{\omega}^{\gamma_1\rho_1}$ given in (171) is in agreement with (105) if we take

$$\hat{e}^{\gamma_1}_{\sigma_1} = \begin{pmatrix} -\frac{1}{2\Delta}\delta^{m_1}_{p_1} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{\Delta}\delta^{s_1}_{n_1} \end{pmatrix} \tag{172}$$

and

$$\omega^{\sigma_1\tau_1} = \begin{pmatrix} \mathbf{0} & \delta^{p_1}_{p_2} \\ -\delta^{s_2}_{s_1} & \mathbf{0} \end{pmatrix}. \tag{173}$$

Consequently, the inverse of $\hat{e}^{\gamma_1}_{\sigma_1}$ of the form (172) reads as

$$\hat{E}^{\sigma_1}_{\tau_1} = \begin{pmatrix} -2\delta^{p_1}_{p_2}\Delta & \mathbf{0} \\ \mathbf{0} & -\delta^{s_2}_{s_1}\Delta \end{pmatrix}. \tag{174}$$

Using (160), (172) and (174), we deduce that relation (107) is automatically verified. Based on formula (104), from (162) and (174) it follows that

$$A^{\alpha_1}_{\alpha_0} = \begin{pmatrix} -Z^{k_1}_{i_1i_2} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{2}Z^{j_1j_2}_{l_1} \end{pmatrix}. \tag{175}$$

We remark that $A^{\alpha_1}_{\alpha_0}$ from (175) is expressed like in (108) for $\sigma^{\alpha_1\beta_1}$ taken as the inverse of (158) and

$$\sigma_{\alpha_0\beta_0} = \begin{pmatrix} \mathbf{0} & -\frac{1}{2}\delta^{i_3}_{[i_1}\delta^{i_4}_{i_2]} \\ -\frac{1}{4}\delta^{j_1}_{[j_3}\delta^{j_2}_{j_4]} & \mathbf{0} \end{pmatrix}. \tag{176}$$

The variables y_{α_1} in the case of the model under investigation are given by

$$y_{\alpha_1} = \begin{pmatrix} \pi_{k_1} \\ A^{l_1} \end{pmatrix}, \tag{177}$$

where A^k is a vector field and π_k its momentum, conjugated in the Poisson bracket induced by (173). Replacing (150), (175) and (177) in the first relation from (130), we find the concrete form of the irreducible constraints $\tilde{\chi}_{\alpha_0} \approx 0$:

$$\tilde{\chi}^{(1)}_{i_1i_2} \equiv -3\partial^{i_3}\pi_{i_3i_1i_2} - \partial_{[i_1}\pi_{i_2]} \approx 0, \tag{178}$$

$$\tilde{\chi}^{(2)j_1j_2} \equiv -\partial_{j_3} A^{j_3j_1j_2} - \tfrac{1}{2}\partial^{[j_1} A^{j_2]} \approx 0. \tag{179}$$

Substituting the second relation from (151) together with (177) in the second formula from (130), we find the irreducible constraints $\tilde{\chi}_{\alpha_2} \approx 0$ for the model under study as

$$\tilde{\chi}^{(1)} \equiv \partial^{k_1}\pi_{k_1} \approx 0, \qquad \tilde{\chi}^{(2)} \equiv \partial_{l_1} A^{l_1} \approx 0. \tag{180}$$

At this stage, we have constructed all the objects entering the structure of the irreducible Dirac bracket (143). *It is essential to remark that the irreducible second-class constraints are local.* If we construct the irreducible Dirac bracket and evaluate the fundamental Dirac brackets among the original variables, then we finally obtain that these are expressed by relations (167) and (168). This completes the analysis of gauge-fixed 3-form gauge fields.

5. Conclusion

To conclude with, in this paper we have exposed an irreducible procedure for approaching systems with second-order reducible second-class constraints. Our strategy includes three main steps. First, we express the Dirac bracket for the reducible system in terms of an invertible matrix. Second, we establish the equality between this Dirac bracket and that corresponding to the intermediate theory, based on the constraints (98). Third, we prove that there exists an irreducible second-class constraint set equivalent with (98) such that the corresponding Dirac brackets coincide. These three steps enforce the fact that the fundamental Dirac brackets with respect to the original variables derived within the irreducible and original reducible settings coincide. Moreover, the newly added variables do not affect the Dirac bracket, so the canonical approach to the initial reducible system can be developed in terms of the Dirac bracket corresponding to the irreducible theory. The general procedure was exemplified on gauge-fixed 3-forms. Our procedure does not spoil other important symmetries of the original system, such as spacetime locality for second-class field theories.

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